Spectrum of Periodic Anderson-Bernoulli Model with Infinitely Many Gaps & Hyperbolic Locus in $SL(2,\mathbb{R})^n$

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In their recent work Avila, Damanik, and Gorodetski showed that a discrete Schrödinger operator with a potential given by a sum of an ergodic potential defined by a dynamical system with a connected phase space, and a random potential has to have a spectrum consisting of a finite number of intervals. This result does not apply to random discrete Schrödinger operators with periodic background. By applying results by Avila, Bocci, and Yoccoz on finitely generated semigroups of $SL(2,\mathbb{R})$, I have constructed a potential that is a sum of a random and periodic ones, such that the spectrum of the corresponding Schrödinger operator consists of the union of an infinite number of disjoint intervals. By examining the geometry of the hyperbolicity locus in $SL(2,\mathbb{R})^n$, one can show the spectrum of such operators must have dense interior.

Main Result

A *discrete Schrödinger operator* $H : l^2 \to l^2$ with a potential $\{P(n) \in \mathbb{R}\}_{n\in\mathbb{Z}}$ is defined by

$$H(\phi)(n) = \phi(n+1) + \phi(n-1) + P(n)\phi(n)$$

We can consider a few different models and their respective spectrum.

$$\sigma(H) = \{E : H - E \text{ does not have a bounded inverse}\}$$

The Anderson Model is an operator with a potential $P(n)$ defined by a sequence of iid random variables, and the spectrum of this model is known to be the Minkowski sum of $[-2, 2]$ with the support of the distribution, making it the union of a finite number of intervals. The spectrum of a discrete Schrödinger operator with a periodic potential is also known to have a spectrum equal to the union of a finite number of intervals. This work studies the spectrum of the periodic Anderson-Bernoulli model, which has a potential defined by a sequence of iid Bernoulli random variables + a periodic background potential.

Theorem 1 (Main Result) *The discrete Schrödinger cocycle with potential defined by the sequence of distributions* $\{\nu(n)\}$

$$\nu(n) = \begin{cases} 9.99B(p_0) + 0 & \textit{if } n \equiv 0 \mod 4 \\ 9.99B(p_1) + 0.9 & \textit{if } n \equiv 1 \mod 4 \\ 9.99B(p_2) - 9.7 & \textit{if } n \equiv 2 \mod 4 \\ 9.99B(p_3) + 2 & \textit{if } n \equiv 3 \mod 4 \end{cases},$$

has a spectrum with infinitely many gaps in the spectrum. The spectrum has dense interior.

Theorem 2 *The periodic Anderson-Bernoulli model with a background potential of period 2 has a well-defined spectrum consisting of at most 4 disjoint intervals.*

Tools

- Define a *transfer matrix* $A_E(x)$ where $E, x \in \mathbb{R}$ by:

$$A_E(x) = \begin{bmatrix} E - x & -1 \\ 1 & 0 \end{bmatrix}.$$

- Given an ergodic discrete Schrödinger operator defined by the dynamical system (Ω, T, μ) and $f : \Omega \to \mathbb{R}$, define a *Schrödinger cocycle* as:

$$(T, A_E \circ f) : (\Omega, \mathbb{R}^2) \to (\Omega, \mathbb{R}^2)$$
$$(\omega, \vec{v}) \mapsto (T(\omega), A_E(f(\omega)) \cdot \vec{v})$$

Propositions

The first proposition equates the respective Schrödinger cocycle being uniformly hyperbolic with a set of matrices being uniformly hyperbolic. The second proposition defines the spectrum in terms of whether or not the cocycle is uniformly hyperbolic.

- **Proposition 1** *Given a discrete Schrödinger operator with random potential plus a background periodic potential of period n defined by $\{\nu(i)\}$, the Schrödinger cocycle with energy E is uniformly hyperbolic if and only if the following set of matrices is uniformly hyperbolic.*

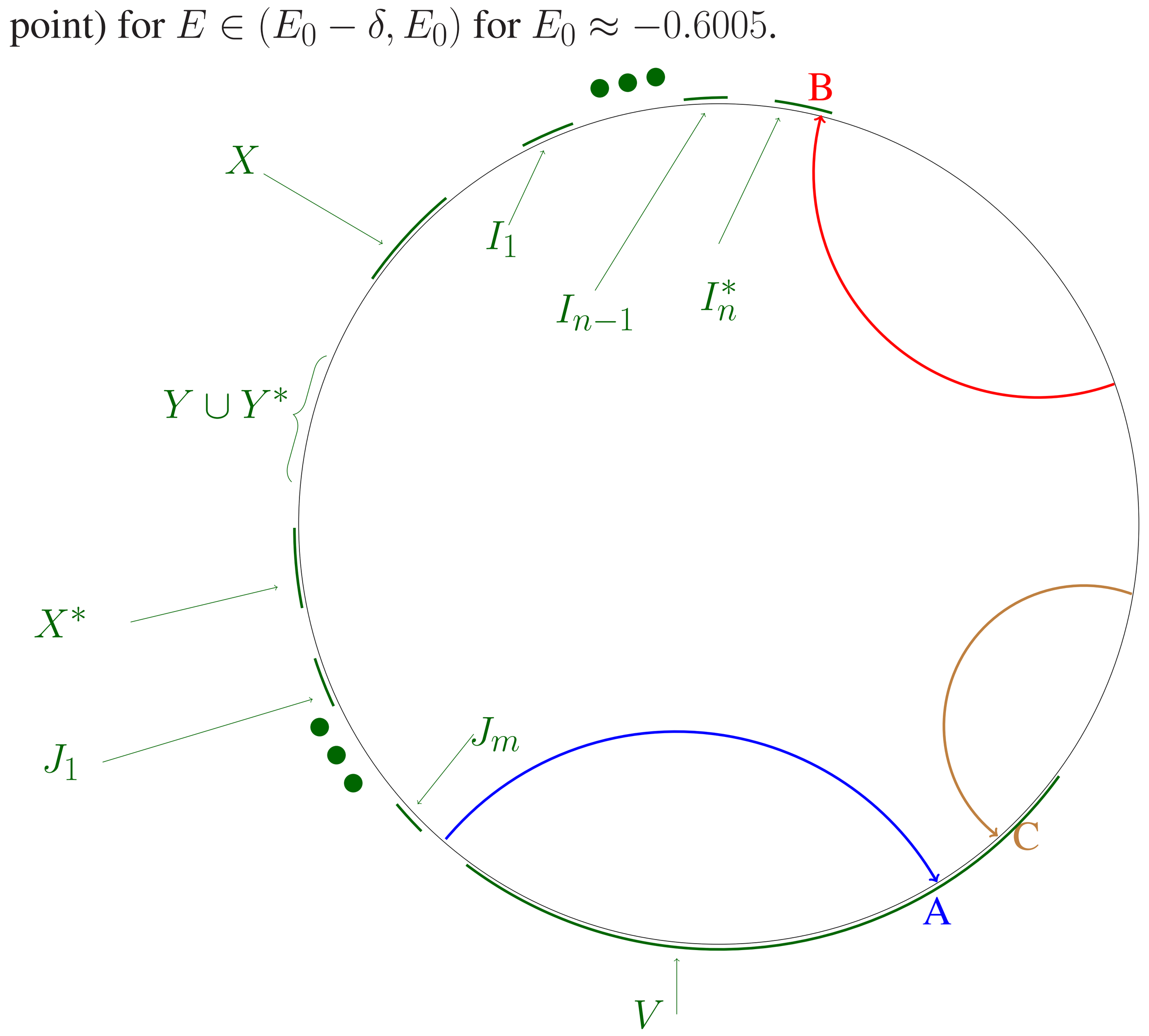
$$\{A_E(\xi_{n-1}) \cdot A_E(\xi_{n-2}) \cdots A_E(\xi_0)\}_{\xi_i \in supp(\nu(i))}$$

- **Theorem 3 (Johnson's Theorem)** *Given ergodic (Ω, T, μ), Ω is a compact metric space, T is a homeomorphism, μ is a T-invariant measure with the support of the measure being Ω, and $f : \Omega \to \mathbb{R}$ is continuous, we have:*

$$\sigma_{as}(H) = \{E : (T, A_E \circ f) \textit{ is not uniformly hyperbolic}\}$$

Approach

The matrices in the set in Proposition 1 are defined by energy E. The set is uniformly hyperbolic over an infinite number of disjoint intervals of E similar to the example from Proposition 4.18 of Avila, Bochi, and Yoccoz's paper. The diagram below shows this by using the cone condition. The circle depicts $\mathbb{RP}^1$, and the curved arrows depict the matrices (resp. Mobius transformations) pointing from the stable eigenvector (resp. repelling point) to the unstable eigenvector (resp. attracting point) for $E \in (E_0 - \delta, E_0)$ for $E_0 \approx -0.6005$.



Uniformly Hyperbolic Sets of Matrices

If a set of matrices $\mathcal{M}$ is uniformly hyperbolic, then we can define the following:

- $\mathcal{C} \subsetneq \mathbb{RP}^1$ consisting of a finite set of closed intervals such that for all $M \in \mathcal{M}$, the corresponding Mobius transformation M' satisfies the condition

$$M'(\mathcal{C}) \subset Int(\mathcal{C})$$

- $\langle \mathcal{M} \rangle = \{M : M = \prod_{i=1}^{n} M_i, \ with \ M_i \in \mathcal{M}\}$
- *Skeleton* $\mathcal{S}_{\mathcal{M}} \subset \mathbb{RP}^1$ is the smallest closed set that is invariant under action of all the matrices of $\mathcal{M}$.
- $\mathcal{M}^{-1} = \{M^{-1} : M \in \mathcal{M}\}$

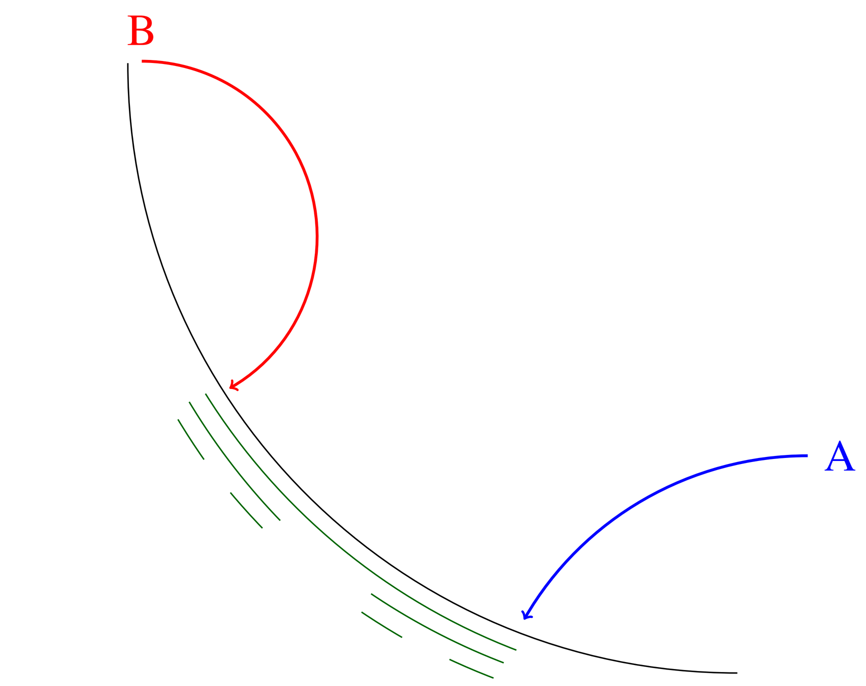
We have that

$$\overline{\{N \cdot u_M : N \in \langle \mathcal{M} \rangle, \ M \in \mathcal{M}\}} = \mathcal{S}$$
$$\overline{\{u_M; M \in \langle \mathcal{M} \rangle\}} = \mathcal{S}$$

Over hyperbolicity locus in $SL(2,\mathbb{R})^n$, the skeleton can be defined and is a subset of every possible cone. By using Theorem 4.1 of Avila, Bochi, and Yoccoz's paper, as points in one of the connected locus approach the boundary, then the distance between $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}_{\mathcal{M}^{-1}}$ goes to zero, or there exists $A \in \mathcal{M}$ such that $A \to Id$.

Skeleton Example

A depiction of what the skeleton look like if the set $\{A, B\}$ has a principal cone.



Assuming the spectral radius of the matrices are sufficiently large, the skeleton is a Cantor set.

The Spectrum Details

Theorem 4 *The spectrum is the closure of the union of infinitely many closed intervals with an interior.*

For random models in question:

$$\Sigma_{\text{AS}} = \overline{\bigcup_{V \text{ is periodic}} \sigma(H_V)}, \qquad (1)$$

where the union ranges over all periodic realizations of the periodic Bernoulli-Anderson model.

Relevant Questions Addressed

Question 1 from Avila, Bochi, and Yoccoz's 2008 paper asks if the boundaries of the connected components of the hyperbolicity locus are disjoint. Details of this work imply a positive answer to this question. Progress in the direction of proving this provides a different proof for Theorem 4.

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